MH370: Finding the Path

Henrik Rydberg

June 3, 2014

Model Basics

In order to explain the recent BFO models [1, 2, 3] of the Burst Frequency Offset (BFO), which accurately reproduces Inmarsat's original BFO graph [4], it is useful to start simple. The basic principle can be found in the V1 model [5], which does not take rate-of-climb into account. This model, as well as other proposals [6] demonstrated to be problematic[7, 8, 9], was first discussed on the TMF blog already in April 2014 [10, 11, 12].

After release of the Inmarsat logs [13, 14], and the nice work to make it machine-readable [15], several BTO models have been proposed [16, 17], and one of them accurately reproduces[18, 19] the elevation angle graph[20].

Preliminaries

We will use a satellite-centered coordinate system here, because it will simplify analysis later on, when looking at the dependency between and along ping rings. We put the point of origin up in space, at an altitude of 35800 kilometers above the imaginary stationary point (0N, 64.5E). We define the y axis as extending from that point towards the center of the earth. The x axis extends towards the east, and the z axis extends towards the north. This creates a right-handed coordinate system. Vectors are denoted in bold, and superscripts are used for both normalized and special scaled vectors. The length of a vector is denoted by the letter itself.

The Effect of Aircraft-Corrected Doppler

Let $\mathbf{p}$ be the position of the aircraft. Let $\mathbf{s}$ be the position of the satellite. Let L be the difference in distance between aircraft-to-satellite and aircraft-to-fixed-satellite,

$$L = |\mathbf{s} - \mathbf{p}| - p. \tag{1}$$

We define the doppler in the usual way, as the rate of change which is positive when objects move towards each other,

$$D = -\frac{f}{c}\frac{dL}{dt}. \tag{2}$$

Here, f is the transmission frequency and c is the speed of light. In our L-band case, $f = 1640 MHz$.

Let $\mathbf{v}_p$ be the velocity of the aircraft, and let $\mathbf{v}_s$ be the velocity of the satellite. Differentiating by parts we obtain

$$D = -\frac{f}{c}\left(\mathbf{v}_p \cdot \nabla_p L + \mathbf{v}_s \cdot \nabla_s L\right). \tag{3}$$

Since $s \ll p$, we can expand L around $\mathbf{s} = \mathbf{0}$. To second order (with typical values of $s/p < 0.03$), the expansion is already pretty good. We get

$$L^{(2)} = -\mathbf{s} \cdot \hat{\mathbf{p}} + s^2/2p. \tag{4}$$

Evaluating Eq.(3) we obtain

$$D^{(2)} = \frac{f}{c}\left(\hat{\mathbf{p}} \cdot \mathbf{v}_s + \bar{\mathbf{s}} \cdot (\mathbf{v}_p - \mathbf{v}_s) - \left(\bar{\mathbf{s}} \cdot \hat{\mathbf{p}} - \bar{s}^2/2\right) \hat{\mathbf{p}} \cdot \mathbf{v}_p\right). \tag{5}$$

Here, $\bar{\mathbf{s}}$ is a conveniently scaled position, $\bar{\mathbf{s}} = \mathbf{s}/p$. Since $\mathbf{s}$ is essentially perpendicular to $\mathbf{p}$, the last term is a very small correction. For clarity, we can thus construct the even simpler first-order approximation,

$$D^{(1)} = \frac{f}{c}\left(\hat{\mathbf{p}} \cdot \mathbf{v}_s + \bar{\mathbf{s}} \cdot (\mathbf{v}_p - \mathbf{v}_s)\right). \tag{6}$$

The first term is the satellite velocity projected onto the aircraft unit vector. The second term is the difference in aircraft and satellite speed, projected onto the scaled satellite vector. Since $\bar{s}$ is a small number, the satellite velocity dominates the expression for most aircraft speeds and headings.

The Burst Time Offset

Similarly to the BFO, the burst time offset (BTO) is related to the difference in propagation delay between the aircraft-to-satellite and fixed-position-to-satellite distances. Let $\mathbf{p}_0$ denote the fixed position (0N, 64.5E). Let H denote said difference,

$$H = |\mathbf{s} - \mathbf{p}| - |\mathbf{s} - \mathbf{p}_0|. \tag{7}$$

We can now express the round-trip time T as

$$T = \frac{2}{c}H + T_0. \tag{8}$$

Here, T_0 is the assumed processing time difference between the aircraft and the nominal terminal.

Finding the Trajectory

With L and H constrained by known data, we can solve the equations. Explicitly,

$$L(t) = L(0) - \frac{c}{f}\int_0^t dt D(t), \tag{9}$$

$$H(t) = \frac{c}{2}(T(t) - T_0). \tag{10}$$

Let r denote the vector from the satellite to the aircraft,

$$\mathbf{r} = \mathbf{p} - \mathbf{s}. \tag{11}$$

We obtain the following expression for the distance,

$$r(t) = H(t) + |\mathbf{s}(t) - \mathbf{p}_0|. \tag{12}$$

These are the dynamic ping rings. Similarly, we obtain the distance between the stationary point and the aircraft as

$$p(t) = r(t) - L(t). \tag{13}$$

These are the stationary ping rings. With both $r(t)$ and $p(t)$ known, the trajectory along the surface of the earth can be uniquely determined. The error at the last ping ring is dominated by the accumulated error in the integration of $L(t)$.

Altitude Changes

The rate of climb is also visible in the BFO. Let $\mathbf{q}$ be the point below $\mathbf{p}$ at sea level. Let $\hat{\mathbf{n}}$ be the surface normal of the earth. Let h denote the altitude of the aircraft. Then,

$$\mathbf{p} = \mathbf{q} + h\hat{\mathbf{n}}. \tag{14}$$

The V2 model can now be stated as

$$L_q = |\mathbf{s} - \mathbf{p}| - q. \tag{15}$$

With $h/p \ll 1$, we can again expand around zero, and this time it is sufficent to go to first order. Hence,

$$q^{(1)} = p - h\hat{\mathbf{p}} \cdot \hat{\mathbf{n}}. \tag{16}$$

Writing L in terms of L_q, we obtain

$$L = L_q^{(1)} - h\hat{\mathbf{p}} \cdot \hat{\mathbf{n}}. \tag{17}$$

The aircraft altitude is thus directly visible in the BFO. In order to obtain the true position p, we need to subtract an estimate of the altitude, projected as above. This is a major correction, because the measured L_ρ is of the order of 20 km, which is on the same scale as a typical altitude of 10 km.

Let $\mathbf{m}$ be the position of the center of the earth, and let R be its (spherical) radius. We can then use the law of cosines to obtain

$$\hat{\mathbf{p}} \cdot \hat{\mathbf{n}} = \frac{p^2 + R^2 - m^2}{2pR}. \tag{18}$$

Here, the difference in length between p and r is of no consequence, so we can write the correction as

$$L = L_q^{(1)} - h\frac{r^2 + R^2 - m^2}{2rR}. \tag{19}$$

References

[1] H. Rydberg. http://www.duncansteel.com/archives/743#comment-3513.

[2] V. Ianello. http://www.duncansteel.com/archives/763#comment-4740.

[3] Y. Fah. http://www.duncansteel.com/archives/751#comment-3850.

[4] www.inmarsat.com/wp-content/uploads/2014/03/Inmarsat-Differential-Doppler-Study.pdf.

[5] R. Cole. http://tmfassociates.com/blog/2014/03/24/understanding-the-satellite-ping-conclusion/#comment-22681.

[6] M. Exner. http://tmfassociates.com/blog/2014/03/24/understanding-the-satellite-ping-conclusion/#comment-22459.

[7] www.inmarsat.com/wp-content/uploads/2014/03/Inmarsat-Differential-Doppler-Study.pdf.

[8] H. Rydberg. http://www.duncansteel.com/archives/743#comment-3485.

[9] H. Rydberg. http://www.duncansteel.com/archives/743#comment-3590.

[10] D. Steel. http://tmfassociates.com/blog/2014/03/24/understanding-the-satellite-ping-conclusion/#comment-20643.

[11] http://tmfassociates.com/blog/2014/03/24/understanding-the-satellite-ping-conclusion/#comment-22607.

[12] http://tmfassociates.com/blog/2014/03/24/understanding-the-satellite-ping-conclusion/#comment-21897.

[13] Inmarsat. http://www.dca.gov.my/mainpage/MH370

[14] D. Steel. http://www.duncansteel.com/archives/806.

[15] P. Sladen. https://github.com/sladen/inmarsat-9m-mro.git.

[16] M. Exner. http://www.duncansteel.com/archives/806#comment-5710.

[17] D. Steel. http://www.duncansteel.com/archives/806#comment-5634.

[18] R. Cole. http://www.duncansteel.com/archives/806#comment-5877.

[19] H. Rydberg. http://www.duncansteel.com/archives/806#comment-6172.

[20] D. Steel. http://www.duncansteel.com/archives/743.